A quasianalyticity property for monogenic solutions of small divisor problems

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Abstract

We discuss the quasianalytic properties of various spaces of functions suitable for one-dimensional small divisor problems. These spaces are formed of functions $\mathscr{C}^1$-holomorphic on certain compact sets K_j of the Riemann sphere (in the Whitney sense), as is the solution of a linear or non-linear small divisor problem when viewed as a function of the multiplier (the intersection of K_j with the unit circle is defined by a Diophantine-type condition, so as to avoid the divergence caused by roots of unity). It turns out that a kind of generalized analytic continuation through the unit circle is possible under suitable conditions on the K_j's.

0 Introduction

Following V. Arnold and M. Herman, and in the same line of research as in [MS03], we consider "monogenic functions" in the sense of Émile Borel with a view to small divisor problems. In these problems of dynamical origin, there is a complex parameter q, called multiplier, which must be kept off the roots of unity in order to solve a functional equation; typically, q is the eigenvalue at a fixed point of a one-dimensional complex map that one wants to linearize and one studies the equation (corresponding to the so-called Siegel problem)

$$h(q, qz) = qG\big(h(q, z)\big) \tag{1}$$

(where $G(z) \in z\mathbb{C}\{z\}$ is given, with $G'(0) = 1$, and $h(q, \,.\,)$ is sought in a Banach space of functions holomorphic in the variable z), or the linearized equation $h(q, qz) - qh(q, z) = qg(z)$ (with $g(z) \in z^2\mathbb{C}\{z\}$ given), or the more complicated non-linear equation corresponding to the conjugacy between a circle map and a rigid rotation with rotation number $\frac{1}{2\pi \mathrm{i}} \log q$ (see equation (2)).

We are interested in the dependence of the solution on the multiplier q. Roots of unity act as resonances, because the coefficients of the solution of the problem are inductively defined by expressions which involve division by $q^k - 1$, $k \geq 1$. On the other hand the case where $|q| = 1$ is particularly interesting from the dynamical point of view. One is thus led to define compact sets K_j of the Riemann sphere $\widehat{\mathbb{C}}$ by removing smaller and smaller neighbourhoods of the roots of unity. It is shown in [He85] and [CM08] for the above-mentioned non-linear problems and in [MS03] for the linear one, that the solution is Whitney smooth on the K_j's, which gives rise to an example of "monogenic" function (the definition is recalled in Section 2). In all the cases we consider, the union of the K_j's on which our monogenic functions are defined contains $\widehat{\mathbb{C}} \setminus \mathbb{S}$, where $\mathbb{S}$ denotes the unit circle, and also a subset of $\mathbb{S}$ defined by an arithmetical condition (Bruno or Diophantine condition); in restriction to $\widehat{\mathbb{C}} \setminus \mathbb{S}$, the functions are analytic in the usual sense.

From the point of view of classical analytic continuation, the unit circle $\mathbb{S}$ appears as a natural boundary, because of the density of the roots of unity, but the question arises whether "monogenic continuation" through $\mathbb{S}$ is possible. A related important issue, as emphasized by M. Herman, is that of quasianalyticity: Is a monogenic function determined by its Taylor series at a point? And indeed the Taylor series is well defined at points of $\mathbb{S}$ corresponding to a Diophantine-type condition, but this series is divergent. At the end of [He85], Herman writes: "we believe that É. Borel (...) wanted his monogenic functions to have quasianalytic properties (i.e. monogenic continuation)," but "the (solution of the) linearized equation does not seem to belong to any quasianalytic class". This is confirmed by our work [MS03] (see also Remark 1 in Section 3).

The question of quasianalyticity can also be raised at each point of $\widehat{\mathbb{C}} \setminus \mathbb{S}$ (where convergent Taylor series are available) and, though easier to answer, it is still non-trivial, because the domain of analyticity is not connected.

Instead of the traditional notion of quasianalyticity, one may consider a weaker property: we shall speak of "$\mathscr{H}^1$-quasianalyticity" whenever the functions are determined by their restriction to any subset of positive linear Hausdorff measure (see Section 1). The subject of the Part A of this article is to prove such a property for spaces of monogenic functions defined on compact sets of $\widehat{\mathbb{C}}$ of a certain kind (Section 3); we shall see in Part B that these spaces are large enough to contain the monogenic functions which appear in small divisor problems, so that we obtain a form of monogenic continuation across the unit circle with respect to the multiplier. More specifically, the small divisor problems considered in Part B are:

– The Siegel problem (1), the solution of which is shown to be monogenic in [CM08], relatively to compact sets K_j described in Section 5; their union intersects $\mathbb{S}$ along a set corresponding to the Bruno condition (optimal for this problem).

– The linearized problem, the solution of which is shown to be monogenic in [MS03], relatively to compact sets described in Section 5.

– The complexified local conjugacy problem of circle maps described under the name Problem (C) in Section 4.

In the last case, one is given a family of maps of the form $\theta \mapsto G_{\alpha,\varepsilon}(\theta) = \theta + \alpha + \varepsilon g(\theta)$ with a holomorphic 1-periodic function g of zero mean-value. The relevant multiplier turns out to be $q = \mathrm{e}^{2\pi \mathrm{i} \alpha}$, while ε is here a small complex parameter. The equation to be solved is

$$u(\theta + \alpha) - u(\theta) + \beta = \varepsilon g\big(\theta + u(\theta)\big), \tag{2}$$

where one looks for $\beta \in \mathbb{C}$ and a holomorphic 1-periodic function u of zero mean-value. As explained in Section 4, this amounts to conjugating $G_{\alpha-\beta,\varepsilon}$ to the rigid rotation $G_{\alpha,0}$. We speak of complexified problem because α, θ and εg are not assumed to be real.

The question of the monogenic regularity of (β, u) with respect to α (or, equivalently, with respect to q) was raised by Arnold in [Ar61] without an answer, due to limitations of the method employed there. Later, in [He85], Herman proved the monogenicity of the solution relatively to compact sets defined by means of a Diophantine condition[1] and contained in a narrow strip $\{|\Im m\,\alpha| < \rho\}$ in the complex domain.

We shall be able to extend Herman's regularity result up to domains K_j of the kind which is required to apply the quasianalyticity result of Part A. Indeed, when complexifying the problem of real circle maps, it may seem natural to focus on a strip for α, which corresponds to a neighbourhood of $\mathbb{S}$ in $\widehat{\mathbb{C}}$ for the multiplier q, but it is important for our quasianalyticity results that the domains for q extend up to 0 and ∞.

As a consequence, we obtain for instance that the solution of any of the three above-mentioned small divisor problems with any given multiplier is (at least theoretically) determined by the solution of the same problem for a small set of values of the multiplier (provided this set has positive linear measure) or by the Taylor series of the solution with respect to the multiplier at any other point of $\widehat{\mathbb{C}} \setminus \mathbb{S}$.

Part A: A quasianalyticity property

1 Various notions of quasianalyticity

In this article we call "non-trivial path" the image of any non-constant continuous map from $[0,1]$ to $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, "Jordan arc" the image of a continuous injective map from $[0,1]$ or $(0,1)$ to $\widehat{\mathbb{C}}$, and "Jordan curve" the image of a continuous injective map from $\mathbb{R}/\mathbb{Z}$ to $\widehat{\mathbb{C}}$. The one-dimensional Hausdorff outer measure in $\mathbb{C}$ will be denoted $\mathscr{H}^1$; we extend it to $\widehat{\mathbb{C}}$ by setting $\mathscr{H}^1(A) = \mathscr{H}^1(A \setminus \{\infty\})$ for any $A \subset \widehat{\mathbb{C}}$ (in fact, what will matter for us will not be the precise value of $\mathscr{H}^1(A)$, but whether it is positive or not).

The following definition is inspired by T. Carleman [Ca26, p.2] and A. Beurling [Be89, p.396] (see also [Ko98b, p.275]).

Definition 1. Let $K' \subset K$ be subsets of $\widehat{\mathbb{C}}$ and E be a linear space of functions on K with values in a complex Banach space.

- A subset γ of K is said to be a *uniqueness set* for E if the only function of E vanishing identically on γ is the function $f \equiv 0$.
- We say that E is *$\mathscr{H}^1$-quasianalytic relatively to K'* if any subset of K' of positive $\mathscr{H}^1$-measure is a uniqueness set for E.

Since every non-trivial path has positive $\mathscr{H}^1$-measure (see e.g. [Fa85, p.29]), $\mathscr{H}^1$-quasianalyticity relatively to K' implies that any non-trivial path contained in K' is a uniqueness set, a property which could be termed *pathwise quasianalyticity relatively to K'*.

As is well-known, if Λ is a Jordan arc, then $\mathscr{H}^1(\Lambda)$ coincides with its length, $|\Lambda|$. When this number is finite (i.e. when the arc avoids ∞ and is rectifiable with respect to the usual distance of $\mathbb{C}$), for any U open in Λ one can define length$(U) =$

[1] We followed quite closely [He85] and sticked to this Diophantine condition, although one could have tried to adapt the results of [Ri99] which deals with the Bruno condition (optimal for this problem) by means of Yoccoz's renormalization method.

$\sum |U_j|$, where the U_j's are its connected components; setting, for any subset A of Λ, $\mathrm{length}_\Lambda(A) = \inf\{\mathrm{length}(U)\,;\ U \text{ open in } \Lambda,\ A \subset U\}$, we then have[2]

$$\mathscr{H}^1(A) = \mathrm{length}_\Lambda(A). \tag{3}$$

Definition 2. Let q_0 be a non-isolated point of $K \subset \widehat{\mathbb{C}}$ and E be a linear space of functions on K with values in a complex Banach space B, such that each function of E admits an asymptotic expansion at q_0. We say that E is *quasianalytic at* q_0 if the only function with zero asymptotic expansion at q_0 is the function $f \equiv 0$.

Recall that a function $f\colon K \to B$ is said to admit an asymptotic expansion at q_0 if there exists a sequence $(a_n)_{n\in\mathbb{N}}$ of elements of B such that, for every $N \in \mathbb{N}$, $(q-q_0)^{-N}\big(f(q) - \sum_{n=0}^{N} a_n(q-q_0)^n\big)$ tends to 0 as $q \to q_0$ with the constraint $q \in K$. The sequence of coefficients is then unique:

$$a_N = \lim_{q\to q_0} (q-q_0)^{-N}\left(f(q) - \sum_{n=0}^{N-1} a_n(q-q_0)^n\right), \qquad N \in \mathbb{N}. \tag{4}$$

This hypothesis is met if f is analytic at q_0, but also when K is closed and f is Whitney-differentiable infinitely many times in the complex sense (i.e. $\mathscr{C}^\infty$-holomorphic, see below) on K.

According to Definition 2, quasianalyticity at q_0 means that the functions are determined by the coefficients a_n of their asymptotic expansions. Observe that this implies that any set $\gamma \subset K$ of which q_0 is a limit point is a uniqueness set for E. Indeed, formula (4) shows that the coefficients of the asymptotic expansion of a function are inductively determined by its restriction to γ.

The usual notion of quasianalyticity (in the sense of Hadamard) is quasianalyticity at every point (see e.g. [Th96]). The latter property is a priori stronger than $\mathscr{H}^1$-quasianalyticity relatively to K (because any set γ of positive $\mathscr{H}^1$-measure has a limit point in it).[3]

If the interior of K, henceforth denoted by $\overset{\circ}{K}$, has several connected components, the pathwise or $\mathscr{H}^1$-quasianalyticity of E relatively to $\overset{\circ}{K}$ imply a form of *coherence*: if two functions of E coincide in one of the connected components of $\overset{\circ}{K}$, then they coincide everywhere; given a function of E, one may also think of its restriction to any of the components as of the "pseudocontinuation" or "generalized analytic continuation" of its restriction to one of them, even though analytic continuation may be impossible (compare with [RS02, pp.18,49]). Similar remarks apply when E is quasianalytic at the points of $\overset{\circ}{K}$.

[2] Indeed, the identity (3) holds for all open subsets of Λ and $\mathscr{H}^1$ is a Borel-regular measure on $\mathbb{C}$ [Fe69, §§2.10.2, 2.10.13]; the measure it induces on Λ is Borel-regular and finite, thus $\mathscr{H}^1(A) = \inf\{\mathscr{H}^1(U)\,;\ U \text{ open in } \Lambda,\ A \subset U\}$ for each $A \subset \Lambda$ [Fe69, §§2.2.2–2.2.3].

[3] Notice however that, for the Denjoy-Carleman classes of an interval of the real line, the notions of $\mathscr{H}^1$-quasianalyticity (or pathwise quasianalyticity) relatively to this interval and Hadamard quasianalyticity coincide—see e.g. [Ca26, p.9]—but this has to do with the one-dimensional character of the interval, whereas we shall rather be interested in compact subsets K of $\widehat{\mathbb{C}}$ not contained in any line.

2 $\mathscr{C}^1$-holomorphic functions and monogenic functions

As in [MS03], we are interested in functions which are $\mathscr{C}^1$-holomorphic on compact sets of the Riemann sphere, i.e. these functions are Whitney-differentiable and satisfy the Cauchy-Riemann equations; equivalently, for a compact set K in $\widehat{\mathbb{C}}$ and a complex Banach space B, we say that $f\colon K \to B$ is $\mathscr{C}^1$-holomorphic if it is continuous and there exists a continuous $f^{(1)}\colon K \to B$ such that: for all $q \in K$ and $\varepsilon > 0$, there exists $\delta > 0$ with

$$q_1, q_2 \in K,\ \ |q_1 - q|, |q_2 - q| \le \delta \ \Rightarrow\ \|f(q_2) - f(q_1) - (q_2 - q_1) f^{(1)}(q)\| \le \varepsilon |q_2 - q_1|$$

(using inversion if $q = \infty$, as usual). We then use the notation $f \in \mathscr{C}^1_{\mathrm{hol}}(K, B)$; the linear space of functions we get can be made a complex Banach space by choosing appropriately a norm $\|\,.\,\|_{\mathscr{C}^1_{\mathrm{hol}}(K,B)}$ (see [MS03, §2.1]). The definition of $\mathscr{C}^\infty$-holomorphic functions on K is in the same vein (op.cit.; [Ri99]).

For the moment we impose no restriction on the compact sets K we consider, but in Section 3 we shall restrict ourselves to very specific ones (see Figure 1).

Definition 3. Suppose $(K_j)_{j\in\mathbb{N}}$ is a monotonic non-decreasing sequence of compact subsets of $\widehat{\mathbb{C}}$ and $(B_j)_{j\in\mathbb{N}}$ is a monotonic non-decreasing sequence of complex Banach spaces with continuous injections $B_j \hookrightarrow B_{j+1}$. The corresponding space of *monogenic functions* is the Fréchet space obtained as the projective limit of Banach spaces

$$\mathscr{M}\big((K_j), (B_j)\big) = \varprojlim \mathcal{A}_J,$$

$$\mathcal{A}_J = \bigcap_{0\le j\le J} \mathscr{C}^1_{\mathrm{hol}}(K_j, B_j), \qquad \|f\|_{\mathcal{A}_J} = \max_{0\le j\le J} \|f_{|K_j}\|_{\mathscr{C}^1_{\mathrm{hol}}(K_j,B_j)}.$$

Indeed, the $\mathcal{A}_J$'s with the continuous injections $\mathcal{A}_J \hookrightarrow \mathcal{A}_{J-1}$ give rise to a projective system and the projective limit $\mathscr{M}\big((K_j), (B_j)\big)$ is a complete topological vector space for the family of semi-norms $\big(\|\,.\,\|_{\mathcal{A}_J}\big)_{J\ge 0}$. As a set, $\mathscr{M}\big((K_j), (B_j)\big)$ consists of all the functions which are defined in $\mathscr{F} = \bigcup_{j\in\mathbb{N}} K_j$ and such that, for every $j \in \mathbb{N}$, the restriction $f_{|K_j}$ belongs to $\mathscr{C}^1_{\mathrm{hol}}(K_j, B_j)$ (this space may depend on the sequence $(K_j)_{j\in\mathbb{N}}$ rather than on $\mathscr{F}$ only).

In [He85] or [MS03], the definition is given with a fixed Banach space $B = B_j$ for all $j \in \mathbb{N}$, in which case $\mathcal{A}_J = \mathscr{C}^1_{\mathrm{hol}}(K_J, B)$. Typically, B is the Hardy space $H^\infty(\mathbb{D}_r)$ consisting of bounded holomorphic functions in a disk $\mathbb{D}_r = \{|z| < r\}$. When applying these ideas to the linear small divisor problem described in Section 4, the drawback of keeping B_j constant as in [MS03] is that the optimal arithmetical condition (see (16) below) cannot be reached: $\mathscr{F}$ is smaller than it could be. Similarly, in the Siegel problem, capturing all the points of the unit circle which satisfy the Bruno condition (11) requires to consider a sequence of decreasing disks. We shall thus take $B_j = H^\infty(\mathbb{D}_{r_j})$ with $r_j \downarrow 0$ in these applications.

In all these cases, each K_j will be a compact arcwise connected subset of $\widehat{\mathbb{C}}$, which intersects the unit circle along a Cantor set avoiding the roots of unity, the interior of which has two connected components, one inside and the other outside the unit circle, while

$$\bigcup_{j\in\mathbb{N}} \mathring{K}_j = \{\,|q| < 1\,\} \cup \{\,|q| > 1\,\} \ \subset\ \mathscr{F} = \bigcup_{j\in\mathbb{N}} K_j \ \subset\ \overline{\mathscr{F}} = \widehat{\mathbb{C}} \tag{5}$$

(both inclusions will be strict).

Obviously, for any compact K and Banach space B, we have the inclusion

$$\mathscr{C}^1_{\mathrm{hol}}(K,B) \subset \mathscr{O}(K,B) = \{\, f\colon K \to B \text{ continuous in } K \text{ and holomorphic in } \overset{\circ}{K}\,\}. \tag{6}$$

In the results of the following section, it is in fact $\mathscr{O}(K,B)$ itself which will be proved to enjoy quasianalyticity properties in certain circumstances, and this will imply similar properties for the smaller space $\mathscr{C}^1_{\mathrm{hol}}(K,B)$. We thus find it worthwhile to mention an inclusion which goes in the reverse direction. Recall that the inner boundary of K is defined as $\partial K \setminus \bigcup \partial U_\ell$ where the U_ℓ's denote the connected components of $\widehat{\mathbb{C}} \setminus K$.

Lemma 1. *Assume that the inner boundary of K is contained in an analytic curve and that the boundary of each connected component U_ℓ of $\widehat{\mathbb{C}} \setminus K$ is a union of rectifiable Jordan curves. Suppose that K contains a compact $\tilde K$ such that any two close enough points q, q' of $\tilde K$ can be joined by a rectifiable path of length $\le c|q-q'|$ inside $\tilde K$, and*

$$q \in \tilde K \ \Rightarrow\ \sum_\ell \int_{\partial U_\ell} \frac{|\mathrm{d}\zeta|}{|\zeta - q|^3} \le C, \tag{7}$$

where c and C are positive constants. Then $\mathscr{O}(K,B) \subset \mathscr{C}^1_{hol}(\tilde K,B)$.

Proof. By Melnikov's theorem [Za68, p.112], the assumption implies that any $f \in \mathscr{O}(K,B)$ is the uniform limit of a sequence of rational functions r_k with poles off K. Given $q \in \tilde K$, the function $\zeta \mapsto f(\zeta)$ and $\zeta \mapsto |\zeta - q|$ are bounded on K, thus there exists $\kappa > 0$ such that $|\zeta - q|^{-1} \le \kappa|\zeta - q|^{-3}$ and $|\zeta - q|^{-2} \le \kappa|\zeta - q|^{-3}$ and we can set

$$f^{(0)}(q) = \frac{1}{2\pi\mathrm{i}} \sum_\ell \int_{\partial U_\ell} \frac{f(\zeta)}{\zeta - q}\,\mathrm{d}\zeta, \quad f^{(1)}(q) = \frac{1}{2\pi\mathrm{i}} \sum_\ell \int_{\partial U_\ell} \frac{f(\zeta)}{(\zeta - q)^2}\,\mathrm{d}\zeta.$$

With a suitable orientation of the ∂U_ℓ's, applying the Cauchy theorem to the rational functions r_k and passing to the limit, we see that $f^{(0)}(q) = f(q)$.

Take now $q, q' \in \tilde K$ close enough one to the other, with a rectifiable path γ joining them inside $\tilde K$. It will be sufficient to show that $A := \|f(q') - f(q) - (q'-q)f^{(1)}(q)\|$ is $O\big((\mathrm{length}(\gamma))^2\big)$. We have

$$f(q') - f(q) - (q'-q)f^{(1)}(q) = \frac{1}{2\pi\mathrm{i}} \sum_\ell \int_{\partial U_\ell} f(\zeta)\mathcal{R}(q,q',\zeta)\,\mathrm{d}\zeta,$$

where

$$\mathcal{R}(q,q',\zeta) = \frac{1}{\zeta - q'} - \frac{1}{\zeta - q} - \frac{q'-q}{(\zeta - q)^2}$$

can also be written $\int_\gamma \frac{2(q'-q_1)}{(\zeta - q_1)^3}\,\mathrm{d}q_1$ (Taylor formula with integral remainder). By Fubini's theorem, we get

$$A \le \frac{C}{\pi} \max|f| \int_\gamma |q' - q_1|\,|\mathrm{d}q_1|$$

and the conclusion follows. $\square$

A similar idea is used in Section 2.5 of [MS03] (see also Remark 2.1 there), where specific compact sets $K^* \subset K$ are defined and satisfy conditions stronger than (7) which imply $\mathscr{C}^1_{\mathrm{hol}}(K,B) \subset \mathscr{C}^\infty_{\mathrm{hol}}(K^*,B)$ (the inner boundaries of the compact sets used in the application to small divisor problems are contained in the unit circle).

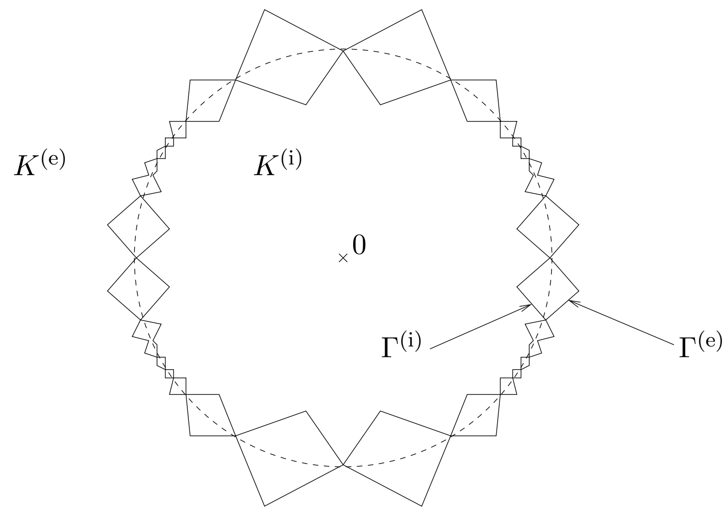

Figure 1: $K = K\big(\Gamma^{(\mathrm{i})}, \Gamma^{(\mathrm{e})}\big)$ is the union of the sets $K^{(\mathrm{i})}$ and $K^{(\mathrm{e})}$ delimited by the internal and external curves $\Gamma^{(\mathrm{i})}$ and $\Gamma^{(\mathrm{e})}$.

3 A quasianalyticity result for $\mathscr{O}(K, B)$ and monogenic functions

The compact sets we are interested in are defined as follows:

Definition 4. We say that $\big(\Gamma^{(\mathrm{i})}, \Gamma^{(\mathrm{e})}\big)$ is a *nested pair* if $\Gamma^{(\mathrm{i})}$ and $\Gamma^{(\mathrm{e})}$ are Jordan curves contained in $\mathbb{C}$ such that

- $\Gamma^{(\mathrm{i})}$ is contained in the closure of the connected component of $\widehat{\mathbb{C}} \setminus \Gamma^{(\mathrm{e})}$ which does not contain ∞,
- $\Gamma^{(\mathrm{i})}$ and $\Gamma^{(\mathrm{e})}$ are rectifiable and $\mathscr{H}^1(\Gamma^{(\mathrm{i})} \cap \Gamma^{(\mathrm{e})}) > 0$.

We then define $K^{(\mathrm{e})}$ to be the closure of the connected component of $\widehat{\mathbb{C}} \setminus \Gamma^{(\mathrm{e})}$ which contains ∞ (delimited by the "external" curve $\Gamma^{(\mathrm{e})}$), $K^{(\mathrm{i})}$ to be the closure of the connected component of $\widehat{\mathbb{C}} \setminus \Gamma^{(\mathrm{i})}$ which does not contain ∞ (delimited by the "internal" curve $\Gamma^{(\mathrm{i})}$), and

$$K\big(\Gamma^{(\mathrm{i})}, \Gamma^{(\mathrm{e})}\big) = K^{(\mathrm{i})} \cup K^{(\mathrm{e})}, \quad \mathcal{I}\big(\Gamma^{(\mathrm{i})}, \Gamma^{(\mathrm{e})}\big) = \Gamma^{(\mathrm{i})} \cap \Gamma^{(\mathrm{e})}. \tag{8}$$

In practice, in the applications considered in Part B, we shall have furthermore

$$\Gamma^{(\mathrm{i})} \subset \overline{\mathbb{D}} \setminus \{0\}, \quad \Gamma^{(\mathrm{e})} \subset \overline{\mathbb{E}} \setminus \{\infty\}, \quad \Gamma^{(\mathrm{i})} \cap \mathbb{S} = \Gamma^{(\mathrm{e})} \cap \mathbb{S}, \tag{9}$$

where

$$\mathbb{D} = \{\, q \in \mathbb{C} \mid |q| < 1 \,\}, \quad \mathbb{E} = \{\, q \in \widehat{\mathbb{C}} \mid |q| > 1 \,\}, \quad \mathbb{S} = \{\, q \in \mathbb{C} \mid |q| = 1 \,\},$$

and the set $\mathcal{I}\big(\Gamma^{(\mathrm{i})}, \Gamma^{(\mathrm{e})}\big) \subset \mathbb{S}$ will be defined by an arithmetical condition which gives it positive Lebesgue measure on the unit circle. Then $\mathcal{I}\big(\Gamma^{(\mathrm{i})}, \Gamma^{(\mathrm{e})}\big)$ coincides with $K^{(\mathrm{i})} \cap \mathbb{S} = \Gamma^{(\mathrm{i})} \cap \mathbb{S}$ and with $K^{(\mathrm{e})} \cap \mathbb{S} = \Gamma^{(\mathrm{e})} \cap \mathbb{S}$ (see Figure 1).

Theorem A. *Let $\big(\Gamma^{(\mathrm{i})}, \Gamma^{(\mathrm{e})}\big)$ be a nested pair and B a complex Banach space. Let $K = K\big(\Gamma^{(\mathrm{i})}, \Gamma^{(\mathrm{e})}\big)$. Then $\mathscr{O}(K, B)$ is $\mathscr{H}^1$-quasianalytic relatively to K and it is also quasianalytic at every point of $\overset{\circ}{K}$.*

Being a smaller space, $\mathscr{C}^1_{\mathrm{hol}}(K,B)$ inherits these quasianalyticity properties. Observe that, by construction, $\overset{\circ}{K}$ has two connected components, $\overset{\circ}{K}{}^{(\mathrm{i})}$ and $\overset{\circ}{K}{}^{(\mathrm{e})}$ (respectively contained in $\mathbb{D}$ and $\mathbb{E}$ when (9) is fulfilled); our functions thus enjoy the aforementioned coherence property, while examples in [MS03] show that the unit circle may be a barrier for the ordinary analytic continuation.

Corollary A. *Let $(B_j)_{j\in\mathbb{N}}$ denote a monotonic non-decreasing sequence of complex Banach spaces with continuous injections. Assume that $\big(\Gamma_j^{(\mathrm{i})},\Gamma_j^{(\mathrm{e})}\big)_{j\in\mathbb{N}}$ is a sequence of nested pairs such that the sequence of compact sets defined by $K_j = K\big(\Gamma_j^{(\mathrm{i})},\Gamma_j^{(\mathrm{e})}\big)$ is monotonic non-decreasing.*

Then the space of monogenic functions $\mathscr{M}\big((K_j),(B_j)\big)$ is $\mathscr{H}^1$-quasianalytic relatively to $\bigcup K_j$ and it is also quasianalytic at every point of $\bigcup \overset{\circ}{K}_j$.

Observe that, under the assumption (5) (which will hold in the applications of Part B), the functions of $\mathscr{M}\big((K_j),(B_j)\big)$ are holomorphic both in $\mathbb{D}$ and in $\mathbb{E}$. They enjoy the aforementioned coherence property: pseudocontinuation is possible through the unit circle.

Remark 1. According to [MS03], for the kind of compact sets $K = K\big(\Gamma^{(\mathrm{i})},\Gamma^{(\mathrm{e})}\big)$ which appear in the small divisor problems of Part B, it is possible to reinforce the arithmetical condition which defines $\mathcal{I} = K\cap\mathbb{S}$ so as to define $\mathcal{I}^* = K^*\cap\mathbb{S}\subset\mathcal{I}$, where $K^* = K\big(\Gamma_*^{(\mathrm{i})},\Gamma_*^{(\mathrm{e})}\big)\subset K$ with a new nested pair $(\Gamma_*^{(\mathrm{i})},\Gamma_*^{(\mathrm{e})})$, in such a way that the functions of $\mathscr{C}^1_{\mathrm{hol}}(K,B)$ admit asymptotic expansions of Gevrey type at the points of $\mathcal{I}^*$. Besides, $\mathscr{C}^1_{\mathrm{hol}}(K,B)\subset\mathscr{C}^\infty_{\mathrm{hol}}(K^*,B)$, as was alluded to at the end of Section 2. One can then raise the question of the (Hadamard) quasianalyticity at the points of K^*, but this is more difficult.

For instance, the points of $\mathbb{S}$ which satisfy the so-called "constant-type" Diophantine condition belong to $\mathcal{I}^*$, and it is shown in [MS03, §3.3] that $\mathscr{C}^1_{\mathrm{hol}}(K,B)$ is not contained in any of the classical Carleman classes quasianalytic at these points—it is in fact the solution itself of the linear small divisor problem that does not belong to these quasianalytic classes.

In [He85], Herman alludes to Borel's studies to determine conditions on the K_j's which ensure the quasianalyticity of the monogenic functions at least at certain points, but they are not fulfilled here (see [Wi93] and [MS03, Remark 2.4]).

Proof of Theorem A. Let $f\in\mathscr{O}(K,B)$; we must infer $f\equiv 0$ from its vanishing on certain subsets of K or from the vanishing of all its derivatives at a given point of $\overset{\circ}{K}$. Without loss of generality, we can take $B=\mathbb{C}$ (because the dual of B separates the points of B and $\ell\circ f\in\mathscr{O}(K,\mathbb{C})$ for every $\ell\in B^*$). Let $U^{(\mathrm{i})}$ denote the connected component of $\widehat{\mathbb{C}}\setminus\Gamma^{(\mathrm{i})}$ which contains 0 (which is nothing but the interior of $K^{(\mathrm{i})}$) and $V^{(\mathrm{e})}$ the connected component of $\widehat{\mathbb{C}}\setminus\Gamma^{(\mathrm{e})}$ which contains ∞ (the interior of $K^{(\mathrm{e})}$).

(a) We first prove that $f_{|\mathcal{I}}\equiv 0 \;\Rightarrow\; f\equiv 0$. The key argument comes from Koosis's proof of Privalov's uniqueness theorem [Ko98a].

By a theorem of Carathéodory, any conformal representation $\varphi^{(\mathrm{i})}\colon\mathbb{D}\to U^{(\mathrm{i})}$ extends to a homeomorphism $\Phi\colon\overline{\mathbb{D}}\to K^{(\mathrm{i})}$. Assuming $f_{|\mathcal{I}}\equiv 0$, we thus have a function $F^{(\mathrm{i})} = f\circ\Phi\in\mathscr{O}(\overline{\mathbb{D}},\mathbb{C})$ which vanishes identically on $\mathcal{J}=\Phi^{-1}(\mathcal{I})\subset\mathbb{S}$. According to a theorem of F. and M. Riesz, the image by Φ of a subset of zero Lebesgue measure of $\mathbb{S}$ has zero arc-length on $\Gamma^{(\mathrm{i})}$ (see [Ko98a, p.54]); now $\mathcal{I}\subset\Gamma^{(\mathrm{i})}$

and, by virtue of (3), $\text{length}_{\Gamma^{(\mathrm{i})}}(\mathcal{I}) = \mathscr{H}^1(\mathcal{I}) > 0$, hence the Lebesgue measure of $\mathcal{J}$ is positive. It follows, by a uniqueness theorem for H^1 functions (see [Ko98a, p.57]), that $F^{(\mathrm{i})} \equiv 0$.

Hence f vanishes identically on $K^{(\mathrm{i})}$. A similar argument with a conformal mapping $\varphi^{(\mathrm{e})}\colon \mathbb{D} \to V^{(\mathrm{e})}$ yields $f \equiv 0$ on $K^{(\mathrm{e})}$.

(b) Suppose now that all the derivatives of f vanish at a point $q_* \in \overset{\circ}{K}$. The principle of analytic continuation yields $f \equiv 0$ on the connected component of q_* in $\overset{\circ}{K}$, which is either $U^{(\mathrm{i})}$ or $V^{(\mathrm{e})}$, and then, by continuity, $f \equiv 0$ on $K^{(\mathrm{i})}$ or $K^{(\mathrm{e})}$ accordingly. In particular, $f \equiv 0$ on $\mathcal{I}$ and, by (a), we get $f \equiv 0$ on the whole of K.

(c) Suppose finally that γ is a subset of K with $\mathscr{H}^1(\gamma) > 0$. By subadditivity, at least one of the sets $\gamma \cap \overset{\circ}{K}$, $\gamma \cap \Gamma^{(\mathrm{i})}$ or $\gamma \cap \Gamma^{(\mathrm{e})}$ must have positive $\mathscr{H}^1$-measure.

– If the first set has positive $\mathscr{H}^1$-measure, then it has an accumulation point and we get $f \equiv 0$ by virtue of (b)(cf. (4) and the observation in the paragraph following it).

– If one of the last two ones, say $\gamma \cap \Gamma^{(\mathrm{i})}$, has positive $\mathscr{H}^1$-measure, then (3) implies that $\text{length}_{\Gamma^{(\mathrm{i})}}(\gamma \cap \Gamma^{(\mathrm{i})}) > 0$ and we can argue as in (a): we take any conformal representation $\varphi^{(\mathrm{i})}\colon \mathbb{D} \to U^{(\mathrm{i})}$, extend it to a homeomorphism $\Phi\colon \overline{\mathbb{D}} \to K^{(\mathrm{i})}$; we have $F^{(\mathrm{i})} = f \circ \Phi$ holomorphic in $\mathbb{D}$, continuous in $\overline{\mathbb{D}}$, vanishing on $\Phi^{-1}(\gamma \cap \Gamma^{(\mathrm{i})}) \subset \mathbb{S}$ which has positive Lebesgue measure (still by F. and M. Riesz's theorem); thus $F^{(\mathrm{i})}$ vanishes identically and so does $f_{|K^{(\mathrm{i})}}$. In particular $f \equiv 0$ on $\mathcal{I}$ and, by the result obtained in (a), $f \equiv 0$ on the whole of K. □

Proof of Corollary A. We show in fact slightly more: the space $\bigcap \mathscr{O}(K_j, B_j)$, which is usually larger than $\mathscr{M}\big((K_j),(B_j)\big)$, is itself $\mathscr{H}^1$-quasianalytic relatively to $\mathscr{F} = \bigcup K_j$ and quasianalytic at the points of

$$\mathscr{F}' = \bigcup \overset{\circ}{K}_j \subset \overset{\circ}{\mathscr{F}}.$$

Let $f \in \bigcap \mathscr{O}(K_j, B_j)$ (this function is thus defined in $\mathscr{F}$), let $\gamma \subset \mathscr{F}$ with $\mathscr{H}^1(\gamma) > 0$ and let $q_* \in \mathscr{F}'$. By subadditivity of $\mathscr{H}^1$, we have $\mathscr{H}^1(\gamma \cap K_j) > 0$ for j large enough. Also, $q_* \in \overset{\circ}{K}_j$ for j large enough.

Assuming that f vanishes on γ, Theorem A thus yields $f \equiv 0$ on K_j for all j large enough, hence on $\mathscr{F}$. The same is true if we assume that the derivatives of f vanish at q_*. □

Remark 2. *(Relation with Privalov's uniqueness theorem.)* Privalov's theorem asserts that a function holomorphic in $\mathbb{D}$ with zero as non-tangential limit at every point of a subset of positive Lebesgue measure of $\mathbb{S}$ must vanish identically.

Assume that the hypotheses of Corollary A are satisfied and let $\mathscr{F} = \bigcup K_j$. Assume moreover that (5) holds and that, for any $\lambda = \mathrm{e}^{2\pi \mathrm{i} x} \in \mathscr{F} \cap \mathbb{S}$ and $c, d > 0$ such that $d < 1$, there exists $j \in \mathbb{N}$ such that the region

$$\{\, q = r\,\mathrm{e}^{2\pi \mathrm{i}\theta} \mid c|\theta - x| \le |r - 1| \le d \,\}$$

is contained in K_j.

These extra assumptions will be met in the small divisor problems of Part B. They imply that a function $f \in \mathscr{M}\big((K_j),(B_j)\big)$ is holomorphic in $\mathbb{D} \cup \mathbb{E}$ and that, at each point $\lambda \in \mathscr{F} \cap \mathbb{S}$, it admits $f(\lambda)$ as non-tangential limit.

In this case, if f vanishes on a set $\gamma \subset \mathscr{F} \cap \mathbb{S}$ of positive Lebesgue measure, one can deduce $f \equiv 0$ on $\mathbb{D}$ directly from Privalov's theorem, and also $f \equiv 0$ on $\mathbb{E}$ by the same theorem (using inversion), hence f vanishes on the whole of $\mathscr{F}$ by continuity.

Part B: Applications to linear and non-linear small divisor problems

4 Introduction to small divisor problems

We now fix the notations for three small divisor problems, to which the results of Part A will be applied in the subsequent sections. They are, by order of increasing complexity, Problem (L), Problem (S) and Problem (C). We begin by presenting the second one, which is the so-called Siegel problem:

Problem (S). *Let $G(z) = z + g(z)$ with $g(z) = \sum_{k\geq 2} g_k z^k \in z^2\mathbb{C}\{z\}$. Study the solution $h(z) = z + \sum_{k\geq 2} h_k z^k$ of the conjugacy equation*

$$h(qz) = qG\big(h(z)\big) \tag{10}$$

as a function of the parameter $q \in \mathbb{C}$.

Equation (10) describes indeed the conjugacy between the germ $z \mapsto q\cdot\big(z+g(z)\big)$ and its linear part $z \mapsto qz$; the parameter q is called the multiplier. It is well known that, when q is not a root of unity, (10) has a unique formal solution h tangent to the identity. The power series $h(z)$ is always convergent when $q \in \mathbb{C}^* \setminus \mathbb{S}$, whereas an arithmetical condition is needed when $q = \mathrm{e}^{2\pi\mathrm{i}\alpha} \in \mathbb{S}$: the so-called Bruno condition [Br72, Yo95], which reads

$$\sum_{k=0}^{\infty} \frac{\log m_{k+1}(\alpha)}{m_k(\alpha)} < \infty, \tag{11}$$

where $\big(m_k(\alpha)\big)_{k\geq 0}$ denotes the sequence of the denominators of the convergents of the irrational real number α (see Appendix A.1). The set

$$\mathcal{I}^{(\mathrm{S})} = \{\, q = \mathrm{e}^{2\pi\mathrm{i}\alpha} \mid \alpha \in \mathbb{R}\setminus\mathbb{Q} \text{ satisfies (11)} \,\} \tag{12}$$

has full measure in $\mathbb{S}$.

Since the solution of (10) depends on q, we shall denote it by $h(q,z)$ instead of $h(z)$. Its coefficients are uniquely determined by induction and are rational functions of q: with the convention $h_1 = 1$, the recurrence formulas are

$$h_k = \frac{1}{q^{k-1}-1} \sum_{j=2}^{k} g_j \sum_{\substack{k_1,\ldots,k_j\geq 1\\ k_1+\cdots+k_j=k}} h_{k_1}\cdots h_{k_j}, \qquad k \geq 2. \tag{13}$$

It is easy to see that $q \mapsto h(q,\,.\,)$ is analytic at each point $q_0 \notin \mathbb{S}$ (with values in a Banach space of holomorphic functions of z which depends on q_0), including the extreme cases $q_0 = 0$ and $q_0 = \infty$ (for which $h_{|q=0}$ is the functional inverse of G and $h_{|q=\infty}$ is reduced to the identity), i.e. it is natural to let q vary in $\widehat{\mathbb{C}}$ in Problem (S).

From the point of view of analyticity with respect to q, we thus get two distinct holomorphic functions $h_{|\mathbb{D}}$ and $h_{|\mathbb{E}}$, but analytic continuation through $\mathbb{S}$ is impossible, at least in the classical sense, because of the small divisors $q^{k-1} - 1$ (each root of unity acts as a resonance, being a pole for infinitely many h_k's). Still, whenever (11) is satisfied, $h(\mathrm{e}^{2\pi\mathrm{i}\alpha}, .)$ is the limit of $h(q, .)$ as $q \to \mathrm{e}^{2\pi\mathrm{i}\alpha}$ non-tangentially (cf. Section 2), as is mentioned in [BMS00, §2.1]. The point here is to go farther than this continuity property by using the results of Part A.

Linearizing the conjugacy equation (10) written as $h(qz) - qh(z) = qg\big(h(z)\big)$ leads to

Problem (L). *Let $g(z) = \sum_{k\geq 2} g_k z^k \in z^2\mathbb{C}\{z\}$. Study the solution*

$$h(z) = h_g(q, z) = z + \sum_{k\geq 2} g_k \frac{z^k}{q^{k-1} - 1} \tag{14}$$

of the "cohomological equation"

$$h(qz) - qh(z) = qg(z) \tag{15}$$

as a function of the parameter $q \in \widehat{\mathbb{C}}$.

The series (14) is uniformly convergent in each compact subset of $(\widehat{\mathbb{C}} \setminus \mathbb{S}) \times \mathbb{D}_R$, where

$$\mathbb{D}_R = \{\, z \in \mathbb{C} \mid |z| < R \,\}$$

is the disk of convergence of $g(z)$. Thus, also in this case we get two holomorphic functions, one for $q \in \mathbb{D}$ and one for $q \in \mathbb{E}$. Also in this case there are non-tangential limits at certain points of the unit circle; this time, the optimal condition for the convergence of the power series in z when $q = \mathrm{e}^{2\pi\mathrm{i}\alpha} \in \mathbb{S}$ is simply:

$$\sup\left\{ \frac{\log m_{k+1}(\alpha)}{m_k(\alpha)},\ k \geq 0 \right\} < \infty \tag{16}$$

(see [MS03, §A.2]); in a way similar to (12), this defines a full-measure subset $\mathcal{I}^{(\mathrm{L})}$ of $\mathbb{S}$, which is a countable union of nowhere dense closed sets, while its complement in $\mathbb{S}$ is a dense G_δ-set with zero s-dimensional Hausdorff measure for all $s > 0$.

The last problem we consider is again non-linear; it can be viewed as a complexification of the problem of local conjugacy of a circle maps:

Problem (C). *Let $G_{\alpha,\varepsilon}(\theta) = \theta + \alpha + \varepsilon g(\theta)$ with $g(\theta) = \sum_{k\in\mathbb{Z}^*} g_k\, \mathrm{e}^{2\pi\mathrm{i}k\theta}$ holomorphic in the annulus $S_R = \{\, \theta \in \mathbb{C}/\mathbb{Z} \mid |\Im m\, \theta| < R \,\}$. Study the solution (β, h), where $\beta \in \mathbb{C}$ and $h : \theta \mapsto \theta + u(\theta)$ with u a 1-periodic holomorphic function of zero mean-value, of the conjugacy equation*

$$G_{\alpha,\varepsilon}\big(h(\theta)\big) - \beta = h(\theta + \alpha) \tag{17}$$

as a function of the parameters $q = \mathrm{e}^{2\pi\mathrm{i}\alpha} \in \widehat{\mathbb{C}}$ and $\varepsilon \in \mathbb{C}$ for small $|\varepsilon|$.

Equation (17) describes the conjugacy between the map $G_{\alpha,\varepsilon} - \beta = G_{\alpha-\beta,\varepsilon}$ and the rigid rotation $G_{\alpha,0} : \theta \mapsto \theta + \alpha$. Observe that the correction β is needed and it is impossible to impose a priori its value since the "rotation number" $\rho(G_{\alpha,\varepsilon})$ need not coincide with α for nonzero ε; in fact, β is implicitly *determined* (modulo 1) by the equation

$$\rho(G_{\alpha-\beta,\varepsilon}) = \alpha \tag{18}$$

as a function of α and ε which is 1-periodic in α. Of course, to speak of complex rotation number, we need a generalization with respect to the classical theory of circle diffeomorphisms (in which $g(\theta)$ is assumed to be real for real values of θ and only real values of α and ε are considered)—see [Ri99] for a geometric insight on this generalization.

Instead of solving first equation (18) and then the conjugacy equation (17), it is possible to obtain directly the pair (β, h) by rewriting the conjugacy equation as $h(\theta+\alpha) - h(\theta) - \alpha + \beta = \varepsilon g\big(h(\theta)\big)$ and defining the operator

$$E_q \ : \ v(\theta) = \sum_{k\in\mathbb{Z}} v_k\, \mathrm{e}^{2\pi \mathrm{i} k\theta} \ \mapsto \ E_q v(\theta) = \sum_{k\in\mathbb{Z}^*} \frac{v_k}{q^k - 1} \mathrm{e}^{2\pi \mathrm{i} k\theta} \tag{19}$$

for $q = \mathrm{e}^{2\pi\mathrm{i}\alpha}$. Indeed, denoting the mean-value of a 1-periodic function by $\langle\,.\,\rangle$, Problem (C) is then equivalent to

$$\beta = \langle v \rangle, \quad h(\theta) = \theta + E_q v(\theta), \tag{20}$$

with v solution of the fixed-point equation

$$v(\theta) = \varepsilon g\big(\theta + E_q v(\theta)\big). \tag{21}$$

In Section 6.3 we shall see that, for $q \in \widehat{\mathbb{C}} \setminus \mathbb{S}$, E_q defines a bounded operator on the Banach space $\mathscr{O}(\overline{S}_{R/2}, \mathbb{C})$ of the holomorphic functions in the annulus $S_{R/2}$ which are continuous on its closure; this will allow us to prove, for $|\varepsilon|$ small enough, the existence of a unique solution (β, h) close to $(0, \mathrm{Id})$, which depends holomorphically on $q \in \widehat{\mathbb{C}} \setminus \mathbb{S}$ and ε.

On the other hand, [He85] deals with the regularity in q of this solution in subsets of $\{\, q = \mathrm{e}^{2\pi\mathrm{i}\alpha} \mid \alpha \in \mathbb{C}/\mathbb{Z},\ |\,\Im m\, \alpha| \le R/100 \,\}$ defined with the help of an arithmetical condition: a constant $\tau \in (0,1)$ being fixed once for all, these sets are defined in such a way that their union intersects $\mathbb{S}$ along a set $\mathcal{I}^{(\mathrm{C})}$ which corresponds to Diophantine numbers α of exponent $2+\tau$, i.e.

$$\sup\left\{ m^{-2-\tau} \left|\alpha - \tfrac{n}{m}\right|^{-1},\ \tfrac{n}{m} \in \mathbb{Q} \right\} < \infty. \tag{22}$$

Complementing Herman's regularity result with the above-mentioned holomorphy result in $\{\, q = \mathrm{e}^{2\pi\mathrm{i}\alpha} \in \widehat{\mathbb{C}} \mid \ |\,\Im m\, \alpha| \ge R/200 \,\}$, we shall be in a position to apply the results of Part A.

Remark 3. Here, in contrast with Problem (S), we do not try to reach the optimal arithmetical condition, which is known to be the Bruno condition (11) as in the Siegel problem, by Risler's result based on renormalization—see [Ri99], [MY02]. We content ourselves with $\mathcal{I}^{(\mathrm{C})}$, which has still full measure in $\mathbb{S}$.

In the following sections, we shall thus recall the results of [MS03] (with some adaptations), [CM08] and [He85] on the dependence on q of the solution of Problems (L), (S) and (C). In each case, a sequence of compact sets $(K_j)_{j\in\mathbb{N}}$ is obtained by removing smaller and smaller open neighbourhoods of the roots of unity, so that:

- The union of the K_j's consists of all the points of $\widehat{\mathbb{C}}$ except the roots of unity and the points $\mathrm{e}^{2\pi\mathrm{i}\alpha} \in \mathbb{S}$ at which (16), resp. (11), resp. (22), fails; in other words, $\bigcup K_j = \mathbb{D} \cup \mathcal{I} \cup \mathbb{E}$ with $\mathcal{I} = \mathcal{I}^{(\mathrm{L})}$, $\mathcal{I}^{(\mathrm{S})}$ or $\mathcal{I}^{(\mathrm{C})}$.
- The map $q \mapsto h(q,\,.\,)$ belongs to $\mathscr{M}\big((K_j),(B_j)\big)$ with $B_j = H^\infty(\mathbb{D}_{r_j},\mathbb{C})$ (the space of bounded holomorphic functions in $\mathbb{D}_{r_j}$ with values in $\mathbb{C}$) for a suitable sequence $(r_j)_{j\in\mathbb{N}}$ in the first two cases, or $B_j = H^\infty\big(\{|\varepsilon| < r_j\}, \mathscr{O}(\overline{S}_{R/2},\mathbb{C})\big)$ in the last case.

We shall then see that the hypotheses of Corollary A stated in Section 3 are fulfilled, which sheds new light on the relationship between $h_{|\mathbb{D}}$ and $h_{|\mathbb{E}}$.

5 Monogenic regularity and quasianalyticity of the solutions of small divisor problems

Definition 5. For any subset A of the real line which is invariant by integer translations, we set

$$A^{\mathbb{C}} = \big\{\, z \in \mathbb{C} \mid\ \exists \alpha_* \in A \ \text{ such that } \ |\Im m\, z| \ge |\alpha_* - \Re e\, z| \,\big\} \tag{23}$$

and

$$K = \operatorname{Exp}\big(A^{\mathbb{C}}\big) \cup \{0,\infty\} \ \subset\ \widehat{\mathbb{C}}, \tag{24}$$

where $\operatorname{Exp}\colon z \in \mathbb{C} \mapsto \mathrm{e}^{2\pi\mathrm{i}z} \in \mathbb{C}^* \subset \widehat{\mathbb{C}}$. The set K is called the *complex multiplier domain* associated with A. Observe that $K \cap \mathbb{S} = \operatorname{Exp}(A)$.

We shall use the following domains:

(5.1) for any $M > \log 3$,

$$A_M^{(\mathrm{L})} = \Big\{\, \alpha \in \mathbb{R}\setminus\mathbb{Q} \mid\ \forall k \in \mathbb{N},\ \frac{\log m_{k+1}(\alpha)}{m_k(\alpha)} \le M \,\Big\} \tag{25}$$

and the corresponding complex multiplier domain $K_M^{(\mathrm{L})}$;

(5.2) for any $M > 0$,

$$A_M^{(\mathrm{S})} = \{\, \alpha \in \mathbb{R}\setminus\mathbb{Q} \mid \mathscr{B}(\alpha) \le M \,\} \tag{26}$$

and the corresponding complex multiplier domain $K_M^{(\mathrm{S})}$, where

$$\mathscr{B}(\alpha) = \sum_{k\ge 0} \frac{\log a_{k+1}(\alpha)}{m_k(\alpha)} \in [0,+\infty], \qquad \alpha \in \mathbb{R}\setminus\mathbb{Q}, \tag{27}$$

$\big(a_k(\alpha)\big)_{k\ge 0}$ denoting the sequence of partial quotients of α (see Appendix A.1);

(5.3) for any $M > 2\zeta(1+\tau)$ (Riemann's zeta function) with a given $\tau \in (0,1)$,

$$A_M^{(\mathrm{C})} = \left\{ \alpha \in \mathbb{R} \setminus \mathbb{Q} \mid \forall \tfrac{n}{m} \in \mathbb{Q},\ m^{-2-\tau} \left|\alpha - \tfrac{n}{m}\right|^{-1} \le M \right\} \tag{28}$$

and the corresponding complex multiplier domain $K_M^{(\mathrm{C})}$.

Remark 4. The function $\mathscr{B}$ is closely related to the classical Bruno series

$$\hat{\mathscr{B}}(\alpha) = \sum_{k \ge 0} \frac{\log m_{k+1}(\alpha)}{m_k(\alpha)}$$

which is involved in the definition (12) of the optimal set $\mathcal{I}^{(\mathrm{S})}$; there exists indeed a constant C such that $0 \le \hat{\mathscr{B}}(\alpha) - \mathscr{B}(\alpha) \le C$ for all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ (see [CM08]). As a result, for any sequence $M_j \uparrow \infty$,

$$\mathbb{S} \cap \bigcup_{j \in \mathbb{N}} K_{M_j}^{(\mathrm{S})} = \{ \mathrm{e}^{2\pi \mathrm{i}\alpha} \mid \alpha \in \mathbb{R} \setminus \mathbb{Q},\ \hat{\mathscr{B}}(\alpha) < \infty \} = \mathcal{I}^{(\mathrm{S})}. \tag{29}$$

In fact, $\bigcup_{j \in \mathbb{N}} K_{M_j}^{(\mathrm{a})} = \mathbb{D} \cup \mathcal{I}^{(\mathrm{a})} \cup \mathbb{E}$ for (a) = (L), (S) or (C).

Theorem B. *Consider Problem* (L) *or* (S) *with a given $g \in H^\infty(\mathbb{D}_R)$, $R > 0$. Then the solution $h : q \mapsto h(q, .)$ belongs to the spaces $\mathscr{C}^1_{hol}\big(K_M^{(\mathrm{L})}, B_M\big)$ or $\mathscr{C}^1_{hol}\big(K_M^{(\mathrm{S})}, B_M\big)$, with $B_M = H^\infty(\mathbb{D}_{R\,\mathrm{e}^{-4M}})$; for any sequence $M_j \uparrow \infty$, the solution thus defines a monogenic function of $\mathscr{M}\big(K_{M_j}^{(\mathrm{L})}, B_{M_j}\big)$ or $\mathscr{M}\big(K_{M_j}^{(\mathrm{S})}, B_{M_j}\big)$.*

As for Problem (C) *with $g \in \mathscr{O}(\overline{S}_R, \mathbb{C})$ of zero mean-value and $\tau \in (0,1)$ given, for each $M > 2\zeta(1+\tau)$ there exists $r = r(M) > 0$ such that*

$$q \in K_M^{(\mathrm{C})},\ |\varepsilon| < r(M) \ \Rightarrow$$
$$\exists \textit{ solution } (\beta, h) = \big(\beta(q,\varepsilon), h(q,\varepsilon)\big) \in \mathbb{C} \times \mathscr{O}(\overline{S}_{R/2}, \mathbb{C}). \tag{30}$$

Moreover, the function $q \mapsto \big(\beta(q, .), h(q, .)\big)$ belongs to the space $\mathscr{C}^1_{hol}\big(K_M^{(\mathrm{C})}, B_M\big)$ with $B_M = H^\infty\big(\{|\varepsilon| < r(M)\}, \mathbb{C} \times \mathscr{O}(\overline{S}_{R/2}, \mathbb{C})\big)$ and thus defines a monogenic function of $\mathscr{M}\big((K_{M_j}^{(\mathrm{C})}), (B_{M_j})\big)$ for any sequence $M_j \uparrow \infty$.

For (a) = (L), (S) *or* (C), *the spaces $\mathscr{O}\big(K_M^{(\mathrm{a})}, B_M\big)$, which contain the spaces $\mathscr{C}^1_{hol}\big(K_M^{(\mathrm{a})}, B_M\big)$, are $\mathscr{H}^1$-quasianalytic relatively to $K_M^{(\mathrm{a})}$ and quasianalytic at the interior points of $K_M^{(\mathrm{a})}$, and the spaces $\mathscr{M}\big((K_{M_j}^{(\mathrm{a})}), (B_{M_j})\big)$ are $\mathscr{H}^1$-quasianalytic relatively to $\mathbb{D} \cup \mathcal{I}^{(\mathrm{a})} \cup \mathbb{E}$ and quasianalytic at the points of $\mathbb{D} \cup \mathbb{E}$.*

The rest of the article is devoted to the proof of Theorem B.

The domains $K_M^{(\mathrm{a})} \subset \widehat{\mathbb{C}}$ are obtained from the arithmetical conditions (25), (26) or (28). The connection with Part A of the article is provided by the following lemma, which will make it possible to apply Theorem A and its corollary to the spaces $\mathscr{O}\big(K_M^{(\mathrm{a})}, B\big)$ or $\mathscr{M}\big((K_{M_j}^{(\mathrm{a})}), (B_{M_j})\big)$.

Lemma 2. *Let $A \subset \mathbb{R}$ be invariant by integer translations and $K \subset \widehat{\mathbb{C}}$ the corresponding complex multiplier domain as in Definition 5. If A is closed and has positive Lebesgue measure, then there exists a nested pair $\big(\Gamma^{(\mathrm{i})}, \Gamma^{(\mathrm{e})}\big)$ such that*

$$K = K\big(\Gamma^{(\mathrm{i})}, \Gamma^{(\mathrm{e})}\big), \quad \mathrm{Exp}(A) = \mathcal{I}\big(\Gamma^{(\mathrm{i})}, \Gamma^{(\mathrm{e})}\big). \tag{31}$$

Proof of Lemma 2. Let $\Phi : x \in \mathbb{R} \mapsto \operatorname{dist}(x, A)$. This is a 1-periodic function, which satisfies (with the notation of Definition 5)

$$A^{\mathbb{C}} = \{\, x + \mathrm{i}y \mid x, y \in \mathbb{R},\ |y| \geq \Phi(x) \,\}.$$

(Indeed, if $x + \mathrm{i}y \in A^{\mathbb{C}}$, then according to (23) there is $\alpha^* \in A$ such that $|y| \geq |\alpha_* - x| \geq \Phi(x)$; conversely, if $|y| \geq \Phi(x)$, just pick $\alpha_* \in A$ such that $\Phi(x) = |\alpha_* - x|$.)

Since Φ is 1-Lipschitz, we have two rectifiable Jordan curves

$$\Gamma^{(\mathrm{i})} = \big\{ \mathrm{e}^{-2\pi\Phi(x)} \cdot \mathrm{e}^{2\pi\mathrm{i}x},\ x \in \mathbb{R} \big\}, \quad \Gamma^{(\mathrm{e})} = \big\{ \mathrm{e}^{2\pi\Phi(x)} \cdot \mathrm{e}^{2\pi\mathrm{i}x},\ x \in \mathbb{R} \big\},$$

which are easily seen to satisfy (31).

□

Notice that if the connected components of $\mathbb{R} \setminus A$ are denoted $(\alpha_\ell, \alpha'_\ell)$, then the connected components of $\mathbb{C} \setminus A^{\mathbb{C}}$ consist of the open diamonds Δ_ℓ with bases $(\alpha_\ell, \alpha'_\ell)$:

$$\Delta_\ell = \big\{\, x + \mathrm{i}y \mid x \in (\alpha_\ell, \alpha'_\ell),\ |y| < \min(x - \alpha_\ell, \alpha'_\ell - y) \,\big\} \tag{32}$$

Remark 5. Problem (C) can be slightly generalized as follows: introducing the notation

$$S_{a,b} = \{\, \theta \in \mathbb{C}/\mathbb{Z} \mid a < \Im m\, \theta < b \,\} \qquad \text{for } -\infty \leq a < b \leq +\infty, \tag{33}$$

we can consider any $G_{\alpha,g} = \mathrm{Id} + \alpha + g$ with g holomorphic on $S_{a,b}$ and small enough, and look for (β, h) such that $G_{\alpha-\beta,g} \circ h = h \circ G_{\alpha,0}$, with $h - \mathrm{Id}$ of zero mean-value and holomorphic in an annulus $S_{a',b'}$ the closure of which is contained in $S_{a,b}$.

Problem (S) now appears as a particular case, via the map $\theta \mapsto E(\theta) = \mathrm{e}^{2\pi\mathrm{i}\theta}$: for any $g \in z^2\mathbb{C}\{z\}$, let

$$\ell_g(z) = \frac{1}{2\pi\mathrm{i}} \log\big(1 + z^{-1}g(z)\big) \in z\mathbb{C}\{z\}$$

and $\tilde{g} = \ell_g \circ E$. Observe that

$$\tilde{g}(\theta) = \sum_{k\geq 1} \tilde{g}_k\, \mathrm{e}^{2\pi\mathrm{i}k\theta}$$

is holomorphic and bounded in $S_{a,+\infty}$ for a large enough (the larger a, the smaller the maximum of $|g|$), and

$$E \circ (\mathrm{Id} + \alpha + \tilde{g}) = \big[q(\mathrm{Id} + g)\big] \circ E,$$

hence linearizing $q(\mathrm{Id} + g)$ by $h \in z + z^2\mathbb{C}\{z\}$ amounts to conjugating $\mathrm{Id} + \alpha + \tilde{g}$ to $\mathrm{Id} + \alpha$ by $\tilde{h} = \mathrm{Id} + \frac{1}{2\pi\mathrm{i}}\big(\log \frac{h}{\mathrm{Id}}\big) \circ E$ holomorphic in $S_{a',+\infty}$ for a certain a', and $\beta = 0$ in this case.

6 Proof of Theorem B

6.1 Case of the cohomological equation

Let $M > \log 3$. It is shown in [MS03, §2.3] that $[0,1] \cap A_M^{(\mathrm{L})}$ has Lebesgue measure $\geq 1 - \frac{2}{\mathrm{e}^M - 1} > 0$, is totally disconnected, closed and perfect; it follows from (43) and (44) that this set consists of points which are "far enough from the rationals", namely

$$\bigcap_{n/m} \{ \alpha \in \mathbb{R} \mid |\alpha - \tfrac{n}{m}| \geq \tfrac{1}{m\,\mathrm{e}^{Mm}} \} \ \subset \ A_M^{(\mathrm{L})} \ \subset \ \bigcap_{n/m} \{ \alpha \in \mathbb{R} \mid |\alpha - \tfrac{n}{m}| > \tfrac{1}{2m\,\mathrm{e}^{Mm}} \}.$$

It is proved in [MS03, §2.4] that the function $q \mapsto h(q,\,.\,)$ which describes the solution of the cohomological equation (15) belongs to $\mathscr{C}^1_{\mathrm{hol}}\big(K_M^{(\mathrm{L})}, H^\infty(\mathbb{D}_r)\big)$ as soon as $0 < r < R\,\mathrm{e}^{-3M}$. The conclusion then follows easily from Lemma 2 applied to $A_M^{(\mathrm{L})}$.

6.2 Case of the Siegel problem

Although the technical proofs of [CM08] depart a lot from those in [MS03], the results for the set $A_M^{(\mathrm{S})}$ and the regularity of the solution of Problem (S) shown there are similar to those for $A_M^{(\mathrm{L})}$ and Problem (L). In particular, [CM08] shows that $A_M^{(\mathrm{S})}$ is a closed subset of $\mathbb{R}$, which is also totally disconnected and perfect. Moreover, we have

Lemma 3. *For every $M > 0$, the set $A_M^{(\mathrm{S})}$ has positive Lebesgue measure in $\mathbb{R}$.*

The proof is postponed to Appendix A.2.

It is proved in [CM08] that the function $q \mapsto h(q,\,.\,)$ which describes the solution of the conjugacy equation (10) belongs to $\mathscr{C}^1_{\mathrm{hol}}\big(K_M^{(\mathrm{S})}, H^\infty(\mathbb{D}_r)\big)$ as soon as $0 < r \leq R\,\mathrm{e}^{-(3+\delta)M}$ for any $\delta > 0$. The conclusion follows easily from Lemma 2 applied to $A_M^{(\mathrm{S})}$.

6.3 Case of the local conjugacy problem of complex maps of the annulus

Let $M > 2\zeta(1+\tau)$. Observe that

$$(0,1) \setminus A_M^{(\mathrm{C})} \ \subset \ \bigcup \big(\tfrac{n}{m} - \tfrac{M^{-1}}{m^{2+\tau}}, \tfrac{n}{m} + \tfrac{M^{-1}}{m^{2+\tau}}\big) \cap (0,1),$$

where the union extends to all $(n,m) \in \mathbb{Z} \times \mathbb{N}^*$ with $(n,m) = 1$; in fact, since $M > 1$, we can restrict it to $0 \leq n \leq m$, thus the Lebesgue measure of this set is

$$\big|(0,1) \setminus A_M^{(\mathrm{C})}\big| \leq 2M^{-1} + \sum_{m \geq 2} \sum_{n=1}^{m-1} 2M^{-1} m^{-2-\tau} < \frac{2\zeta(1+\tau)}{M}.$$

As a consequence $A_M^{(\mathrm{C})}$ is closed subset of positive Lebesgue measure of $\mathbb{R}$ (and $\mathcal{I}^{(\mathrm{C})}$ has full measure).

As alluded to at the end of Section 5, [He85] studies the regularity of the solution of Problem (C) in

$$\tilde{K}_M^{(\mathrm{C})} = K_M^{(\mathrm{C})} \cap \{ q = \mathrm{e}^{2\pi\mathrm{i}\alpha}, |\Im m\, \alpha| \leq R/100 \}.$$

It is proved there that there exists $r = r(M) > 0$ for which (30) holds with $K_M^{(\mathrm{C})}$ replaced by $\tilde{K}_M^{(\mathrm{C})}$, and that the function $q \mapsto \big(\beta(q,\,.\,), h(q,\,.\,)\big)$ belongs to the space $\mathscr{C}^1_{\mathrm{hol}}\big(\tilde{K}_M^{(\mathrm{C})}, B_M\big)$ with

$$B_M = H^\infty\big(\{|\varepsilon| < r(M)\}, \mathbb{C} \times \mathscr{O}(\overline{S}_{R/2}, \mathbb{C})\big).$$

To conclude, it is sufficient to extend this regularity property from $\tilde{K}_M^{(\mathrm{C})}$ to $K_M^{(\mathrm{C})}$ and to apply Lemma 2 to $A_M^{(\mathrm{C})}$. If we use the rephrasing of Problem (C) as equations (20)–(21), the conclusion thus follows from

Lemma 4. *Let* $\Lambda > 0$*,* $\Omega = \{\, q = \mathrm{e}^{2\pi\mathrm{i}\alpha}, |\Im m\, \alpha| > \Lambda \,\}$*,* $B = \mathscr{O}\big(\overline{S}_{R/2}, \mathbb{C}\big)$ *and* $\mathcal{E} = 2 + (\mathrm{e}^{2\pi R} - 1)^{-1} + \frac{1}{2}\sinh^{-2}(\pi\Lambda)$*. Then:*

(i) For each $q \in \Omega$*,* E_q *is a bounded linear operator of* B*:* $\|E_q v\| \le \mathcal{E}\|v\|$*, and, given* $r' > 0$ *and a function* $(q,\varepsilon) \mapsto v_{q,\varepsilon} \in B$ *holomorphic for* $q \in \Omega$ *and* $|\varepsilon| < r'$*, the function* $(q,\varepsilon) \mapsto E_q v_{q,\varepsilon} \in B$ *is holomorphic too.*

(ii) There exists $r' > 0$ *such that, for all* $q \in \Omega$ *and* ε *such that* $|\varepsilon| < r'$*, there exists a unique solution* v *of equation (21) in* B *close to* 0*. Moreover, this solution* v *is a bounded holomorphic function of* (q,ε)*.*

Proof. As a preliminary, we introduce the following notation: for $k \in \mathbb{Z}$, e_k will denote the function $\theta \mapsto \mathrm{e}^{2\pi\mathrm{i}k\theta}$, and if $v = \sum_{k\in\mathbb{Z}} v_k e_k \in B$,

$$\Pi^+ v = \sum_{k\ge1} v_k e_k, \qquad \Pi^- v = \sum_{k\ge1} v_{-k} e_{-k}, \qquad \Pi_k v = v_k e_k, \qquad k \in \mathbb{Z}.$$

Since the Fourier coefficients of v can be computed as

$$v_k = \int_{\mathrm{i}\rho}^{1+\mathrm{i}\rho} v(\theta)\, \mathrm{e}^{-2\pi\mathrm{i}k\theta}\, \mathrm{d}\theta$$

for any $\rho \in \big[-\frac{R}{2}, \frac{R}{2}\big]$, it follows that

$$|v_k| \le \mathrm{e}^{-\pi|k|R}\|v\|, \qquad k \in \mathbb{Z}, \tag{34}$$

hence, with the notation (33), $\Pi^\pm v$ is holomorphic in $S^+ = S_{-R/2,+\infty}$ or $S^- = S_{-\infty,R/2}$ and is in fact a function of $z^\pm = \mathrm{e}^{\pm2\pi\mathrm{i}\theta}$ holomorphic for $|z^\pm| < \mathrm{e}^{\pi R}$. More than this, one has

$$\begin{aligned} &\Pi^\pm v \text{ extends continuously to } \overline{S}^\pm, \\ &|\Pi^\pm v(\theta)| \le \Big(2 + \frac{1}{\mathrm{e}^{2\pi R} - 1}\Big)\|v\| \ \text{ for } \theta \in \overline{S}^\pm. \end{aligned} \tag{35}$$

Indeed, in the case of the '+' sign for instance, $\Pi^+ v = v - \Pi_0 v - \Pi^- v$ where $v - \Pi_0 v$ is holomorphic in S^+, continuous in $\overline{S}^+$ and bounded by $2\|v\|$, while $\Pi^- v$ is holomorphic in any neighbourhood $\mathcal{N}$ of $\{\Im m\,\theta = -R/2\}$ contained in S^-, hence this representation of $\Pi^+ v$ gives a continuous extension to $\mathcal{N} \cap \overline{S}^+$; by the maximal modulus principle, the maximum of $|\Pi^+ v|$ is attained for $|z^+| = \mathrm{e}^{\pi R}$, it is thus equal to $\max\big\{|\Pi^+ v(\theta)|,\ \Im m\,\theta = -R/2\big\} \le 2\|v\| + \max\big\{|\Pi^- v(\theta)|,\ \Im m\,\theta = -R/2\big\}$ and

(35) follows from $\Im m\, \theta = -R/2 \Rightarrow |\mathrm{e}^{-2\pi \mathrm{i} k\theta}| = \mathrm{e}^{-\pi k R}$ and from (34) applied to $|v_{-k}|$ for $k \geq 1$.

We can rephrase (34) and (35) as statements about bounded linear operators:

$$\Pi_k, \Pi^{\pm} \in \mathcal{L}(B), \quad \|\Pi_k\| \leq 1, \quad \|\Pi^{\pm}\| \leq 2 + \frac{1}{\mathrm{e}^{2\pi R} - 1}. \tag{36}$$

(i) Assume $q = \mathrm{e}^{2\pi \mathrm{i}\alpha} \in \Omega$ with $|q| < 1$, i.e. $\Im m\, \alpha > \Lambda$ and $|q| < \mathrm{e}^{-2\pi\Lambda}$ (we would argue in a symmetric way in the case $|q| > 1$), and let $v \in B$. Writing $\frac{1}{q^k - 1} = -1 - \frac{q^k}{1-q^k}$ and $\frac{1}{q^{-k}-1} = \frac{q^k}{1-q^k}$ for $k \geq 1$, we get

$$E_q v = -\Pi^+ v + \sum_{k \geq 1} \frac{q^k}{1-q^k}(-\Pi_k + \Pi_{-k})v \tag{37}$$

On the one hand, $|1-q^k| \geq 1 - |q^k| \geq 1 - \mathrm{e}^{-2\pi\Lambda}$ and $|q^k| \leq \mathrm{e}^{-2\pi k\Lambda}$, on the other hand $-\Pi_k + \Pi_{-k} \in \mathcal{L}(B)$ has operator norm ≤ 2 by (36), hence (37) yields a representation of E_q as an absolutely convergent series of bounded linear operators and

$$\|E_q\| \leq \|\Pi^+\| + \sum_{k\geq 1} \frac{\mathrm{e}^{-2\pi k\Lambda}}{1 - \mathrm{e}^{-2\pi\Lambda}} \| - \Pi_k + \Pi_{-k}\| \leq 2 + \frac{1}{\mathrm{e}^{2\pi R} - 1} + \frac{2\mathrm{e}^{-2\pi\Lambda}}{(1 - \mathrm{e}^{-2\pi\Lambda})^2},$$

which was the desired bound. Moreover, the operators Π^+ and $-\Pi_k + \Pi_{-k}$ are independent of (q, ε), the coefficients $\frac{q^k}{1-q^k}$ are holomorphic functions of q and the above convergence was uniform in q, hence we obtain the holomorphic dependence on (q, ε).

(ii) Let $C = \max_{\overline{S}_R} |g|$ and $r' = \frac{R}{8\mathcal{E}C}$. We shall prove the statement with this value of r' by means of the contraction principle.

For any $q \in \Omega$ and $\varepsilon \in \mathbb{C}$ such that $|\varepsilon| < r'$, we define a map $\mathcal{F}_{q,\varepsilon}$ on the ball $\mathscr{V} = \{\, v \in B \mid \|v\| \leq r'C \,\}$ by the formula

$$\mathcal{F}_{q,\varepsilon}(v) \;:\; \theta \in \overline{S}_{R/2} \;\; \mapsto \;\; \varepsilon g\big(\theta + E_q v(\theta)\big) \qquad \text{for } v \in \mathscr{V}.$$

The function $\mathcal{F}_{q,\varepsilon}(v)$ is well-defined when $v \in \mathscr{V}$ because, for every $\theta \in \overline{S}_{R/2}$, $|E_q v(\theta)| \leq \mathcal{E}\|v\| \leq R/8$, hence

$$\theta + E_q v(\theta) \in S_{3R/4}. \tag{38}$$

This function clearly belongs to B, with $\|\mathcal{F}_{q,\varepsilon}(v)\| \leq |\varepsilon| C \leq r'C$. Therefore, $\mathcal{F}_{q,\varepsilon}$ is a self-map of $\mathscr{V}$.

Suppose $v_1, v_2 \in \mathscr{V}$. For each $\theta \in \overline{S}_{R/2}$, (38) and the fact that $\max_{\overline{S}_{3R/4}} |g'| \leq 4C/R$ imply that

$$|\mathcal{F}_{q,\varepsilon}(v_2)(\theta) - \mathcal{F}_{q,\varepsilon}(v_1)(\theta)| \leq \frac{4C\mathcal{E}|\varepsilon|}{R}\|v_2 - v_1\| \leq \frac{1}{2}\|v_2 - v_1\|.$$

We thus get a unique fixed point for $\mathcal{F}_{q,\varepsilon}$, i.e. a unique solution of (21), in $\mathscr{V}$. This solution can be written as the sum of the series

$$\mathcal{F}_{q,\varepsilon}(0) + \sum_{k\geq 0} \big(\mathcal{F}^{k+1}_{q,\varepsilon}(0) - \mathcal{F}^{k}_{q,\varepsilon}(0)\big)$$

which is absolutely convergent in B.

Let us check that this fixed point depends holomorphically on (q,ε) for $q \in \Omega$ and $|\varepsilon| < r'$. Since $\|\mathcal{F}^{k+1}_{q,\varepsilon}(0) - \mathcal{F}^{k}_{q,\varepsilon}(0)\| \le \frac{1}{2^k}\|\mathcal{F}_{q,\varepsilon}(0)\|$, it is sufficient to check that the functions $(q,\varepsilon) \mapsto \mathcal{F}^{k}_{q,\varepsilon}(0)$ are holomorphic. This follows by induction from the fact that, if $(q,\varepsilon) \mapsto v_{q,\varepsilon} \in \mathscr{V}$ is holomorphic for $q \in \Omega$ and $|\varepsilon| < r'$, then $(q,\varepsilon) \mapsto \mathcal{F}_{q,\varepsilon}(v_{q,\varepsilon})$ is holomorphic too (proof: since B is a Banach algebra, $\mathcal{F}_{q,\varepsilon}(v_{q,\varepsilon})$ can be written as the sum of the uniformly convergent series $\varepsilon \sum_{\ell\ge 0} g_\ell (E_q v_{q,\varepsilon})^\ell$, where $g_\ell = \frac{1}{\ell!} g^{(\ell)}{}_{|\overline{S}_{R/2}}$, $\|g_\ell\| \le C(2/R)^\ell$, $\|(E_q v_{q,\varepsilon})^\ell\| \le (\mathcal{E} r' C)^\ell$).

□

A Appendix

A.1 Continued fractions

We indicate here our notations and a few facts that we use in Part B, referring the reader e.g. to [HW79] for an exposition of the theory.

Given $x \in \mathbb{R}\setminus\mathbb{Q}$, we set $a_0 = [x] \in \mathbb{Z}$, $\xi_0 = x - [x] \in (0,1)$, and inductively $a_{k+1} = [\xi_k^{-1}] \in \mathbb{N}^*$, $\xi_{k+1} = \xi_k^{-1} - [\xi_k^{-1}] \in (0,1)$, hence

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_k + \xi_k}}}.$$

Dropping ξ_k in the last expression, we get a rational number called the kth *convergent* of x and denoted by $[a_0, a_1, \ldots, a_k]$. We denote by $\frac{n_k}{m_k}$ the reduced expression of this rational.

The a_k's are called *partial quotients* of x; notice that, for $k \ge 1$, they are all positive integers. We sometimes write $a_k(x)$, $n_k(x)$, $m_k(x)$, considering the partial quotients and the convergents as functions of $x \in \mathbb{R}\setminus\mathbb{Q}$.

Numerators and denominators of the convergents can be obtained from the recursive formulas

$$n_k = a_k n_{k-1} + n_{k-2}, \quad m_k = a_k m_{k-1} + m_{k-2}, \tag{39}$$

with initial conditions $n_{-1} = 1$, $n_{-2} = 0$, $m_{-1} = 0$, $m_{-2} = 1$. They satisfy

$$m_k n_{k-1} - n_k m_{k-1} = (-1)^k, \tag{40}$$

$$x = \frac{n_k + \xi_k n_{k-1}}{m_k + \xi_k m_{k-1}}. \tag{41}$$

The formulas (39) imply that $(m_k)_{k\ge 1}$ is an increasing sequence of positive integers bounded from below by the Fibonacci numbers:

$$m_k \ge F_{k+1} \ge \Phi^{k-1}, \quad F_k = \frac{\Phi^k + (-1)^{k+1}\varphi^k}{\sqrt{5}}, \qquad k \ge 0, \tag{42}$$

with $\varphi = \frac{\sqrt{5}-1}{2} \in (0,1)$ and $\Phi = 1/\varphi = 1 + \varphi$ (golden ratio).

The convergents n_k/m_k converge to x at least geometrically; more precisely,

$$\frac{1}{2m_{k+1}} < |m_k x - n_k| < \frac{1}{m_{k+1}}, \qquad k \geq 1, \tag{43}$$

$\frac{n_k}{m_k} < x < \frac{n_{k+1}}{m_{k+1}}$ for even k (reverse inequalities for odd k), $\left|\frac{n_{k+1}}{m_{k+1}} - \frac{n_k}{m_k}\right| = \frac{1}{m_k m_{k+1}}$. According to the classical *law of best approximation*, if a rational $\frac{n}{m}$ does not belong to the sequence of convergents and if $k \in \mathbb{N}^*$, then

$$m \leq m_{k+1} \quad \Rightarrow \quad |mx - n| > |m_k x - n_k|. \tag{44}$$

As a partial converse to (43), we mention the fact that if a rational n/m satisfies $|mx - n| < \frac{1}{2m}$, then it necessarily belongs to the sequence of convergents of x.

A.2 Proof of Lemma 3

For $\tau \geq 0$, we introduce the Diophantine numbers of $(0,1)$ of exponent $2+\tau$:

$$\mathrm{DC}_{\gamma,\tau} = \left\{ x \in (0,1) \mid \forall \tfrac{n}{m} \in \mathbb{Q} \cap (0,1),\ \left|x - \frac{n}{m}\right| \geq \frac{\gamma}{m^{2+\tau}} \right\}, \qquad \gamma > 0$$

(the smaller γ, the larger $\mathrm{DC}_{\gamma,\tau}$). The set of "constant-type points" of $(0,1)$ can be defined as $\bigcup_{\gamma>0} \mathrm{DC}_{\gamma,2}$, but it won't be of any use for us here because it has zero measure, whereas it is well-known that the larger sets $\bigcup_{\gamma>0} \mathrm{DC}_{\gamma,\tau}$ have full measure for all $\tau > 0$ (see Section 6.3; $\mathrm{DC}_{\gamma,\tau}$ is essentially $A^{(\mathrm{S})}_{\gamma^{-1}} \cap (0,1)$). It follows from (43) that all these numbers satisfy the Bruno condition (11), which we saw is equivalent to $\mathscr{B}(x) < \infty$. For every $M > 0$, we shall find a positive measure subset of them for which $\mathscr{B}(x) \leq M$. The starting point is the observation (obvious from the definition (27)) that

$$a_1(x) = \cdots = a_k(x) = 1 \quad \Rightarrow \quad \mathscr{B}(x) = \sum_{\ell \geq k} \frac{\log a_{\ell+1}(x)}{m_\ell(x)}.$$

Given $k \in \mathbb{N}^*$ and $a_1, \ldots, a_k \in \mathbb{N}^*$, we set $\frac{n_\ell}{m_\ell} = [0, a_1, \ldots, a_\ell]$ for $0 \leq \ell \leq k$; the corresponding "interval of rank k" (see [Kh64]) is then defined as

$$I(a_1, \ldots, a_k) = \left\{ \frac{n_k + \zeta n_{k-1}}{m_k + \zeta m_{k-1}},\ \zeta \in (0,1) \right\} = \left| \begin{array}{ll} \left(\frac{n_k}{m_k}, \frac{n_k + n_{k-1}}{m_k + m_{k-1}}\right) & \text{if } k \text{ is even} \\ \left(\frac{n_k + n_{k-1}}{m_k + m_{k-1}}, \frac{n_k}{m_k}\right) & \text{if } k \text{ is odd.} \end{array} \right.$$

Then $I(a_1, \ldots, a_k) \setminus \mathbb{Q} = \{ x \in (0,1) \setminus \mathbb{Q} \mid a_\ell(x) = a_\ell \text{ for } 1 \leq \ell \leq k \}$ (compare with (41)).

The case where $a_\ell = 1$ for $1 \leq \ell \leq k$ is related to the golden ratio conjugate $\varphi = [0, 1, 1, \ldots]$, for which

$$n_\ell(\varphi) = F_\ell, \quad m_\ell(\varphi) = F_{\ell+1}, \qquad \ell \in \mathbb{N}.$$

We then abbreviate the previous notation:

$$I_k = I(\underbrace{1, 1, \ldots, 1}_{k \text{ times}}),$$

which is an open interval of length $|I_k| = \frac{1}{F_{k+1} F_{k+2}}$ (from now on, we simply denote by $|\,.\,|$ the Lebesgue measure on $\mathbb{R}$).

With these preliminaries, Lemma 3 follows from the more precise

Lemma 5. *(i) For each $M > 0$, $\tau \geq 0$ and $\gamma > 0$, there exists $\bar{k} \in \mathbb{N}^*$ such that*

$$k \geq \bar{k} \;\Rightarrow\; \mathrm{DC}_{\gamma,\tau} \cap I_k \subset A_M^{(\mathrm{S})}.$$

(ii) There exists $\gamma^* > 0$ such that, for $\tau \geq 1$ and $0 < \gamma < \gamma^*$,

$$k \geq 2 \;\Rightarrow\; |\mathrm{DC}_{\gamma,\tau} \cap I_k| > \Big(1 - \frac{\gamma}{\gamma^*}\Big) |I_k| .$$

Proof of (i). Let $M > 0$, $\tau \geq 0$, $\gamma > 0$ and $k \in \mathbb{N}^*$.

If $x \in \mathrm{DC}_{\gamma,\tau}$ and $\ell \in \mathbb{N}^*$, then $\frac{\gamma}{m_\ell^{1+\tau}} \leq |m_\ell x - n_\ell| < \frac{1}{m_{\ell+1}}$ by (43) (denoting by $\frac{n_1}{m_1}, \frac{n_2}{m_2}, \dots$ the convergents of x), and $m_{\ell+1} = a_{\ell+1} m_\ell + m_{\ell-1} > a_{\ell+1} m_\ell$, hence

$$a_{\ell+1} < \frac{m_\ell^\tau}{\gamma}.$$

If we assume moreover $x \in I_k$, then $a_1 = \cdots = a_k = 1$ and

$$\mathscr{B}(x) < \log(\gamma^{-1}) \sum_{\ell \geq k} \frac{1}{m_\ell} + \tau \sum_{\ell \geq k} \frac{\log m_\ell}{m_\ell} < \log(\gamma^{-1}) \sum_{\ell \geq k} \frac{1}{m_\ell} + \tau \sum_{\ell \geq k} \frac{2}{\sqrt{m_\ell}}.$$

The inequalities (42) thus yield $\mathscr{B}(x) < \sum\limits_{\ell \geq k} \Big(\log(\gamma^{-1}) \varphi^{\ell-1} + 2\tau \varphi^{\frac{\ell-1}{2}} \Big)$, which can be made $\leq M$ by choosing k large enough since the series is convergent.

Proof of (ii). Let $\tau \geq 1$, $k \geq 2$ and suppose $0 < \gamma < \frac{1}{3}$ to begin with. We have

$$I_k \setminus \mathrm{DC}_{\gamma,\tau} = \bigcup_{n/m \in \mathbb{Q} \cap (0,1)} J_{n/m} \cap I_k, \quad \text{with } J_{n/m} = \Big(\frac{n}{m} - \frac{\gamma}{m^{2+\tau}}, \frac{n}{m} + \frac{\gamma}{m^{2+\tau}} \Big).$$

Our goal is to ensure $|I_k \setminus \mathrm{DC}_{\gamma,\tau}| < \frac{\gamma}{\gamma^*} |I_k|$ for suitable γ^*. Obviously,

$$I_k \setminus \mathrm{DC}_{\gamma,\tau} \subset \bigcup_{n/m \in \mathbb{Q}_{\gamma,\tau,k}} J_{n/m},$$

$$\text{with } \mathbb{Q}_{\gamma,\tau,k} = \big\{ \tfrac{n}{m} \in \mathbb{Q} \cap (0,1) \mid J_{n/m} \cap I_k \neq \emptyset \big\}. \quad (45)$$

From now on, we shall denote by $\frac{n_1}{m_1}, \frac{n_2}{m_2}, \dots$ the convergents of φ, specifying the argument only when referring to a point possibly different from φ; thus, $\frac{n_\ell}{m_\ell} \equiv \frac{F_\ell}{F_{\ell+1}}$. We first establish that any $n/m \in \mathbb{Q}_{\gamma,\tau,k}$ has $m \geq m_k$.

Indeed, for such a rational (which we suppose written in least terms), we may choose an irrational x in the non-empty open interval $I_k \cap J_{n/m}$, for which

$$\Big| x - \frac{n}{m} \Big| < \frac{\gamma}{m^{2+\tau}} < \frac{1}{2m^2}.$$

The "partial converse to (43)" of the end of Appendix A.1 yields $\ell \in \mathbb{N}$ such that $\frac{n}{m} = \frac{n_\ell(x)}{m_\ell(x)}$. Let us check that ℓ cannot be $\leq k-2$ by contradiction: this would lead to

$$\frac{n_\ell(x)}{m_\ell(x)} = \frac{n_\ell}{m_\ell} \quad \text{and} \quad \frac{n_{\ell+2}(x)}{m_{\ell+2}(x)} = \frac{n_{\ell+2}}{m_{\ell+2}}$$

(since $x \in I_k$ and $\ell + 2 \le k$), with both of these rationals on the same side of x, whence

$$\left|x - \tfrac{n}{m}\right| > \left|\frac{n_{\ell+2}}{m_{\ell+2}} - \frac{n_\ell}{m_\ell}\right| = \frac{1}{m_\ell m_{\ell+2}}$$

(the last identity results from (40) applied twice), and since

$$\frac{\gamma}{m^{2+\tau}} = \frac{\gamma}{m_\ell^{2+\tau}} > \left|x - \tfrac{n}{m}\right|,$$

we would get $\gamma > \frac{m_\ell^{1+\tau}}{m_{\ell+2}}$, which is easily seen to be $\ge 1/3$.

In fact, ℓ cannot be equal to $k-1$ either, for this would lead to $\frac{n_k+n_{k-1}}{m_k+m_{k-1}}$ lying between $\frac{n}{m} = \frac{n_{k-1}}{m_{k-1}} \notin I_k$ and the points of I_k (because this point is the $(k+1)$th convergent of φ and thus lies on the same side of φ as $\frac{n_{k-1}}{m_{k-1}}$), hence

$$\left|\frac{n_{k-1}}{m_{k-1}} - \tfrac{n_k+n_{k-1}}{m_k+m_{k-1}}\right| < \left|x - \frac{n_{k-1}}{m_{k-1}}\right| < \frac{\gamma}{m_{k-1}^{2+\tau}},$$

where the left-hand side is

$$\frac{1}{m_{k-1}(m_{k-1}+m_k)} > \frac{1}{2m_{k-1}m_k},$$

whence $\gamma > \frac{m_{k-1}^{1+\tau}}{2m_k} > 1$, a contradiction. Hence, $\ell \ge k$ and $m = m_\ell(x) \ge m_k(x) = m_k$ (because $x \in I_k$).

Now, let p_m denote, for any $m \in \mathbb{N}^*$, the number of integers n such that $n/m \in \mathbb{Q}_{\gamma,\tau,k}$. We just saw that $p_m = 0$ for $m < m_k$; we now prove that $p_m < 3 + m\,|I_k|$.

Suppose indeed $p_m \ge 3$ for a given m (which is necessarily $\ge m_k$), and let n^- and n^+ denote the minimal and maximal numerators such that $n/m \in \mathbb{Q}_{\gamma,\tau,k}$; thus $p_m = n^+ - n^- + 1$. As the intervals $J_{n/m}$ are disjoint (they are separated by a distance $\frac{1}{m} - \frac{2\gamma}{m^{2+\tau}} > \frac{1}{m} - \frac{1}{m^2} > 0$), we see that both $\frac{n^-+1}{m}$ and $\frac{n^+-1}{m}$ belong to I_k, hence $\frac{n^+-n^--2}{m} < |I_k|$, which yields the desired inequality.

Therefore, (45) implies that

$$|I_k \setminus \mathrm{DC}_{\gamma,\tau}| \le \sum_{m \ge m_k} \frac{2\gamma p_m}{m^{2+\tau}} < \big(6Z_{1+\tau}(m_k) + 2\,|I_k|\,Z_\tau(m_k)\big)\gamma,$$

with the notation

$$Z_\alpha(N) = \sum_{m \ge N} \frac{1}{m^{1+\alpha}} < \frac{1}{\alpha(N-1)^\alpha} \quad \text{for } N \ge 2.$$

Observe that $|I_k| = \frac{1}{m_k(m_k+m_{k-1})} > \frac{1}{2m_k^2}$, hence $\frac{Z_{1+\tau}(m_k)}{|I_k|} < 2m_k^2 Z_2(m_k) < 4$, while $Z_\tau(m_k) \le Z_1(2) \le 1$, thus $\gamma^* = 1/26$ will do. $\square$

Remark. A simple adaptation of the above proof of *(ii)* yields the following more general result: for each $\bar\gamma > 0$ and $\bar\tau \ge 0$, for each $\bar x = [0, a_1, a_2, \dots] \in \mathrm{DC}_{\bar\gamma,\bar\tau}$ and $\tau \ge \max(1, \bar\tau)$,

$$0 < \gamma < \min\Big(\frac{1}{26}, \frac{\bar\gamma}{2}\Big) \text{ and } k \ge 2 \;\Rightarrow\; |\mathrm{DC}_{\gamma,\tau} \cap I(a_1,\dots,a_k)| > \Big(1 - \frac{\gamma}{\gamma^*}\Big)\,|I(a_1,\dots,a_k)|.$$

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